ON THE ABSORPTION OF NONLINEAR WAVES BY DISPERSING MEDIA

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The effect of dissipative processes on the propagation of nonlinear waves in dispersing media is analyzed here. It is explained in what manner the wave attenuation depends on the nonlinearity parameter and on the character of the dissipation mechanism. Equations are derived which describe the propagation of a solitary pulse or so-called soliton in such a medium.

1. As is well known, wave processes in slightly nonlinear dissipationless media can sometimes be described approximately by the Korteweg-deVries equation

$$u_t + uu_x + \beta u_{xxx} = 0 \tag{1.1}$$

Thus, (1.1) describes the propagation of surface waves on "shallow" water [1, 2], of acoustic and magnetohydrodynamic waves in plasma [2], of electromagnetic waves in nonlinear transmission lines, etc. Steady-state solutions to Eq. (1.1) – cnoidal waves – have been throughly analyzed (see, e.g., [3]). A special class of such solutions are solitary waves (solitons), which play an important role in the theory of transient "fission" processes [2].

The effect of dissipation processes on nonlinear waves has been studied only in a few individual cases. The propagation of a solitary wave in plasma, for instance, was analyzed in [4] taking into account a Landau attenuation.

Here we will consider the effect of various kinds of dissipation processes on the propagation of nonlinear waves in dispersing media as a function of the nonlinearity parameter characterizing the profile of the steady-state solution to (1.1).

2. When the dissipation is sufficiently weak, Eq. (1.1) assumes the approximate form

$$u_t + uu_x + \beta u_{xxx} + \alpha u - \delta u_{xx} = 0 \tag{2.1}$$

Thus, the term δu_{xx} in the case of surface waves and plasma waves accounts for the viscosity of the medium (δ represents here the kinematic viscosity). The term αu accounts for the friction between the fluid and the ground or the air [5]. These two terms correspond respectively to the high-frequency and the low-frequency losses in nonlinear lines transmitting electromagnetic waves. Depending on the specific characteristics of a system, one or the other dissipation mechanism may be predominant. For a solution which is periodic in space one can easily obtain from (2.1) a series of integral relations of the "conservation law" type [2]. Integrating (2.1) over the period Λ, for instance, we have

$$\int_0^\Lambda u(x, t)\,dx = \text{const}\, e^{-\alpha t} \tag{2.2}$$

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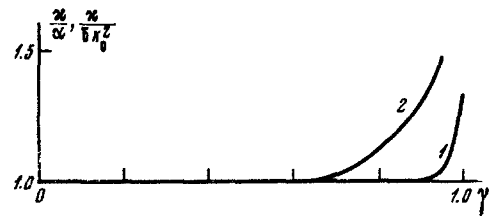

Fig. 1

In this way, a decrease in the "momentum" of a wave is due to low-frequency losses only. Analogous equations are obtained for the energy $\int u^2 dx$, etc.

3. Equation (2.1) will now be solved approximately under the assumption that the dissipation terms are small, so that locally the wave is almost cnoidal, and the solution will be written in the convenient form [6]:

$$u = \frac{12\beta k^2}{\pi} K(\gamma) \frac{\partial}{\partial\theta} Z\left[\frac{K(\gamma)\theta}{\pi};\gamma\right], \qquad \theta = \omega t - kx \tag{3.1}$$
$$\omega = \frac{4\beta k^3}{\pi^2} K^2(\gamma)\left[2-\gamma-3\frac{E(\gamma)}{K(\gamma)}\right]$$

Here Z is the Jacobi zeta-function with the θ-period 2 and with the mean value zero, $E(\gamma)$ and $K(\gamma)$ are complete elliptic integrals with the modulus $\sqrt{\gamma}$, and ω and k are the frequency and the wave number, respectively. Actually, γ is a parameter defining the wave nonlinearity (as $\gamma \to 0$, the wave becomes harmonic; $\gamma = 1$ corresponds to a solitary wave or so-called soliton). The parameter γ also determines the wave amplitude

$$A = u_+ - u_- = \frac{12\beta k^2}{\pi^2}\gamma K^2(\gamma) \tag{3.2}$$

For further analysis it will be convenient to transform Eq. (2.1): with a change of variables

$$\Phi_x = u, \qquad H = u_x - {}^1/_2\delta u \beta^{-1} \tag{3.3}$$

Equation (2.1) can be easily written in the Lagrange form of the second kind:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \Phi_t} + \frac{\partial}{\partial x}\frac{\partial L}{\partial \Phi_x} = -\frac{\partial R}{\partial \Phi_x}, \qquad \frac{\partial}{\partial x}\frac{\partial L}{\partial H_x} - \frac{\partial L}{\partial H} = -\frac{\partial R}{\partial H_x} \tag{3.4}$$

where L is the Lagrangian (the Lagrange function density)

$$L = {}^1/_2\Phi_x\Phi_t + {}^1/_6\Phi_x^3 + \beta H_x\Phi_x + {}^1/_2\beta H^2 - {}^1/_8\delta^2\beta^{-1}\Phi_x^2 \tag{3.5}$$

and R is the Rayleigh function density

$$R = {}^1/_2\alpha\Phi_x^2 - {}^1/_2\delta H_x\Phi_x \tag{3.6}$$

Because energy is dissipated in the system, the solution to (3.1) is strictly not valid, but with small values of α and δ the wave locally approaches a cnoidal one whose envelopes are slow functions of the space coordinates and of time. The equations for the variable amplitude $A(x, t)$, frequency $\omega(x, t) = \theta_t$, and wave number $k(x, t) = -\theta_x$ will be derived using the generalized variational method in terms of averages. According to this principle, the Lagrange function and the Rayleigh function densities must be averaged over a period of the quasi-stationary solution (3.1) and then one must write down the corresponding Lagrange equations of the second kind in the generalized "coordinates" A, θ. (For conservative systems such an approach was first proposed in [7]). The equations of the envelopes are

$$\frac{\partial}{\partial t}\frac{\partial\langle L\rangle}{\partial\omega} - \frac{\partial}{\partial x}\frac{\partial\langle L\rangle}{\partial k} = \frac{\partial\langle R\rangle}{\partial k}, \qquad \frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0 \tag{3.7}$$

The second of Eqs. (3.7) is a consequence of expressing ω and k in terms of θ. Here

$$\langle L\rangle = -{}^1/_2\omega k\langle\Phi_\theta^2\rangle + \beta k^3\langle\Phi_\theta H_\theta\rangle + {}^1/_2\beta\langle H^2\rangle - {}^1/_8\beta^{-1}\delta^2k^2\langle\Phi_\theta^2\rangle \tag{3.8}$$

$$\langle R\rangle = {}^1/_2\alpha k^2\langle\Phi_\theta^2\rangle - {}^1/_2\delta k^2\langle\Phi_\theta H_\theta\rangle \tag{3.9}$$

Equations (3.7) must be supplemented with boundary or with initial conditions. To be specific, we will consider the initial-value problem, i.e., we will find the variation of parameters in the solution which at $t = 0$ correspond to (3.1) with certain given initial values of A_0, k_0, ω_0, γ_0.* Obviously, the space period does not change with time when $t > 0$ ($k \equiv k_0$). The other parameters are functions of time only. Under these conditions, then, (3.7) reduces to

$$\frac{dY_1(\gamma)}{dt} = -2\alpha Y_1(\gamma) - 2\delta k_0^2 Y_2(\gamma) \tag{3.10}$$

where

$$\begin{aligned} Y_1(\gamma) &= K^2\langle Z_\theta^2\rangle = \frac{K^4}{\pi^2}\left[\frac{4-2\gamma}{3}\frac{E}{K} - \frac{1-\gamma}{3} - \frac{E^2}{K^2}\right] \\ Y_2(\gamma) &= K^2\langle Z_{\theta\theta}^2\rangle = \frac{4K^6}{15\pi^4}\left[2(\gamma^2-\gamma+1)\frac{E}{K} - (1-\gamma)(2-\gamma)\right] \end{aligned} \tag{3.11}$$

While determining γ from (3.10), we will at the same time find all the wave characteristics: its amplitude and frequency. The value of γ is of interest for other reasons too: this parameter defines the effective width of the wave's spectrum (i.e., its degree of nonsinusoidality).

The energy dissipation is described conveniently in terms of the absorption coefficient

$$\varkappa = -\frac{d}{dt}\ln\frac{A(t)}{A_0} \tag{3.12}$$

Using (3.2) and (3.10), we have for $\varkappa$,

$$\varkappa = 2(\alpha Y_1 + \delta k_0^2 Y_2)\frac{d\ln(\gamma K^2)/d\gamma}{dY_1/d\gamma} \tag{3.13}$$

The curves of $\varkappa/\alpha$ (for $\delta = 0$) and of $\varkappa/\delta k_0^2$ (for $\alpha = 0$) as functions of γ are shown in Fig. 1. We note that the absorption coefficient $\varkappa$ changes rapidly only near $\gamma = 1$. This has to do with the fact that elliptic functions differ from trigonometric functions substantially only near $\gamma \approx 1$ (see, e.g. [3]) and, therefore, a quasilinear approximation is possible here over a wide range of $\varkappa$.

For small and for large values of γ it is easy to derive simple asymptotic expressions for $\varkappa$ as well as for $\gamma(t)$ and $A(t)$.

When $\gamma \ll 1$ (the wave is almost sinusoidal), then

$$\begin{gathered} Y_1 \approx Y_2 \approx {}^1/_{128}\,\pi^2\gamma^2, \qquad K(\gamma) \approx {}^1/_2\,\pi \\ \varkappa = \alpha + \delta k_0^2, \quad \gamma(t) = \gamma_0 e^{-\varkappa t}, \quad A(t) = A_0 e^{-\varkappa t} \end{gathered} \tag{3.14}$$

It makes sense that this final result can also be obtained from (2.1) without difficulty by neglecting there the nonlinear term uu_x.

When $1-\gamma \ll 1$ (the wave becomes almost a sequence of weakly interfering solitons), however, then

$$Y_1 \approx \frac{2}{3\pi^2}K^3, \quad Y_2 \approx \frac{8}{15\pi^4}K^5, \quad K(\gamma) \approx \frac{1}{2}\ln\frac{16}{1-\gamma}$$

and we have

$$\varkappa = \frac{4}{3}\alpha + \frac{4}{45}\beta^{-1}\delta A, \quad A(t) = \frac{A_0 e^{-4/_3\,\alpha t}}{1 + {}^1/_{15}\,\delta A_0\alpha^{-1}\beta^{-1}(1 - e^{-4/_3\,\alpha t})} \tag{3.15}$$

*We will note that, since the Korteweg-deVries equation describes wave processes in slightly nonlinear systems, the solution to the boundary-value problem can be arrived at from the solution to the initial-value problem by changing variable t to z/c (c is the linear propagation velocity of the wave [2]; in (1.1) x is the "running" coordinate $x = z - ct$).

It is evident that, when $\delta = 0$, the amplitude of solitons decreases exponentially but faster than it would according to linear theory.

If $\alpha = 0$, then

$$A(t) = \frac{A_0}{1 + {}^4/_{45}\,\delta\beta^{-1}A_0 t} \quad (3.16)$$

The attenuation characteristic is qualitatively different than in the linear case. After the elapse of a long time interval, the attenuation is determined by the dispersion parameter β and the "viscosity" δ only, and it does not depend on the initial pulse amplitude*

$$A(t) \approx \frac{45\,\beta}{4\delta t}, \quad t \to \infty \quad (3.17)$$

The following is to be said concerning the calculation of the absorption coefficient. In the case of slightly nonlinear media, some authors (e.g. of [8]), prefer to determine $\varkappa$ as a sum of partial (with respect to the wave number) absorption coefficients. According to this method, the soliton absorption coefficient would be $\alpha + 0.065\delta\beta^{-1}$ A, i.e., approximately 30% smaller than according to Eq. (3.15) where the nonlinearity is taken into account.

We note, in conclusion, that existing theories agree well with results of the experiment in which the propagation of solitary radio waves along nonlinear transmission lines was studied.

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Note Added in Proof. Equations (3.15) and (3.16) were also obtained by E. Ott and R. N. Sudan in "Damping solitary waves," Phys. of Fluids, 13, No. 6 (1970).

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*In plasma with a Landau attenuation the soliton amplitude decreases as t^{-2} while at $t \to \infty$ the attenuation is also independent of the initial amplitude [4]. An analogous situation exists in the case of shock waves in a viscous medium [8].